Ejercicios Relatividad General. Capítulo 57

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Dado

$$g^{\alpha\beta}\delta\Gamma^{\mu}_{\lambda\sigma} = \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\left(\partial_\sigma\delta g_{\lambda\nu} + \partial_\lambda\delta g_{\sigma\nu} - \partial_\nu\delta g_{\lambda\sigma}\right)$$

Tenemos que encontrar una expresión para $J^\mu = g^{\alpha\beta}\delta\Gamma^{\mu}_{\alpha\beta} - g^{\beta\mu}\delta\Gamma^{\lambda}_{\lambda\beta}$

$$g^{\alpha\beta}\delta\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\left(\partial_\beta\delta g_{\alpha\nu} + \partial_\alpha\delta g_{\beta\nu} - \partial_\nu\delta g_{\alpha\beta}\right)$$

$$g^{\beta\mu}\delta\Gamma^{\lambda}_{\lambda\beta} = \frac{1}{2}g^{\lambda\nu}g^{\beta\mu}\left(\partial_\lambda\delta g_{\beta\nu} + \partial_\beta\delta g_{\lambda\nu} - \partial_\nu\delta g_{\lambda\beta}\right)$$

Renombramos los índices mudos en esta última expresión de la siguiente forma:

$$\beta \to \nu, \qquad \lambda \to \alpha, \qquad \nu \to \beta$$

Y usando que la métrica es un tensor simétrico; $g_{\mu\nu} = g_{\nu\mu}$

$$g^{\mu\nu}\delta\Gamma^{\alpha}_{\alpha\nu} = \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\left(\partial_\alpha\delta g_{\beta\nu} + \partial_\nu\delta g_{\alpha\beta} - \partial_\beta\delta g_{\alpha\nu}\right)$$

Por lo tanto la resta es simplemente

$$\begin{aligned}
J^\mu &= g^{\alpha\beta}\delta\Gamma^{\mu}_{\alpha\beta} - g^{\mu\nu}\delta\Gamma^{\alpha}_{\alpha\nu} \\
&= \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\left(\underbrace{\partial_\beta\delta g_{\alpha\nu}}_{1} + \underbrace{\partial_\alpha\delta g_{\beta\nu}}_{2} - \underbrace{\partial_\nu\delta g_{\alpha\beta}}_{3} - \underbrace{\partial_\alpha\delta g_{\beta\nu}}_{2} - \underbrace{\partial_\nu\delta g_{\alpha\beta}}_{3} + \underbrace{\partial_\beta\delta g_{\alpha\nu}}_{1}\right) \\
&= \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}\left(2\partial_\beta\delta g_{\alpha\nu} - 2\partial_\nu\delta g_{\alpha\beta}\right) = g^{\mu\nu}g^{\alpha\beta}\left(\partial_\beta\delta g_{\alpha\nu} - \partial_\nu\delta g_{\alpha\beta}\right)
\end{aligned}$$

De nuevo, usando que la métrica es simétrica (y, por lo tanto, podemos intercambiar $\alpha \leftrightarrow \beta$ dento del paréntesis) nos queda la expresión

$$\boxed{J^\mu = g^{\mu\nu}g^{\alpha\beta}\left(\partial_\alpha\delta g_{\nu\beta} - \partial_\nu\delta g_{\alpha\beta}\right)}$$